Transitivity of Infinite-Dimensional Extensions of Anosov Diffeomorphisms

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Preliminaries

Necessary background theory from dynamical systems: transitive maps, Anosov diffeomorphisms, extension and group extensions.

Goals

- We are interested in studying group extensions of Anosov diffeomorphisms.
- Explicity, we have $T : X \to X$ Anosov, $f : X \to G$, get the extension

$$T_f : (x, g) \mapsto (Tx, g + f(x))$$

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- Specifically, concerned with which cocycles give transitive extensions? Stably transitive?
- Starting point: the groups $\mathbb{R}^n$.
- What are clearly necessary conditions for transitivity of a $\mathbb{R}^n$ cocycle?

Positive Livsic Theorem

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- We want an easier way to check whether this obstruction is satisifed.
- Let

Definition

P_f is the set of sums of f over periodic orbits of T,
$\left\{\sum_{i=0}^{n-1} f(T^i(x)) | T^n(x) = x\right\}$

Positive Livsic Theorem

Then we have

Theorem (Positive Livsic Theorem:)

Let X be a Riemannian manifold, $T : X \to X$ a transitive Anosov diffeomorphism, and $f : X \to \mathbb{R}$ a Hölder cocycle. If $P_f \subset \mathbb{R}^+ = \{x \in \mathbb{R} | x \geq 0\}$, then f is cohomologous to a Hölder cocycle with image contained in $\mathbb{R}^+$.

- Corollary: for extensions $f : X \to \mathbb{R}^n$,

Theorem

If $\exists v \in \mathbb{R}^n, P_f \subset H_v = \{x \in \mathbb{R} | \langle v, x \rangle \geq 0\}$, then f is cohomologous to a Hölder cocycle with image contained in H_v.

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Definition

A map $f : X \to \mathbb{R}^n$ is called inseparable if its periodic data lie on both sides of any hyperplane, i.e. if $\forall 0 \neq v \in \mathbb{R}^n, P_f \nsubseteq H_v$.

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- We know this condition is necessary for f to be transitive. Is it sufficient?

Previous theorem

- Answer: **Yes!**

Theorem (Nitica and Pollicott, 2005)

If $T : X \to X$ Anosov, X infranil, $f : X \to \mathbb{R}^n$ Hölder cocycle, then f is transitive if and only if it is inseparable.

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- Answer: **Yes!**

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- Our problem: Can this be generalized for an infinite-dimensional real vector space instead of $\mathbb{R}^n$?

Our theorem

- We'll work with the infinite-dimensional space $\mathbb{R}^\omega = \prod_{i=1}^{\infty} \mathbb{R}$. Topological product of countably many copies of $\mathbb{R}$; sequences $(x_i)_j \to_j (y_i)$ iff each x_i converges to y_i as $j \to \infty$, i.e. "pointwise convergence".

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- What should inseparability mean for cocycles $f : X \to \mathbb{R}^\omega$?

Definition

f is inseparable if for any continuous linear functional $L : \mathbb{R}^\omega \to \mathbb{R}, L(P_f) \not\subseteq \mathbb{R}^+$.

- Actually the same as the condition $\pi_n(P_f)$ cannot lie on one side of a hyperplane in $\mathbb{R}^n$, where π_n is projection onto the first n coordinates.

Proposition

Let $L : \mathbb{R}^\omega \to \mathbb{R}$ be a continuous linear functional. Then for some n, $L = L \circ \pi_n$.

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Proposition

Let $L : \mathbb{R}^\omega \to \mathbb{R}$ be a continuous linear functional. Then for some n, $L = L \circ \pi_n$.

- We arrive at the statement of our main result:

Theorem

Let X be an infranilmanifold, $T : X \to X$ an Anosov diffeomorphism, and $f : X \to \mathbb{R}^\omega$ a Hölder cocycle. Then f is transitive if and only if it is inseparable.

Brief sketch of proof: write the set of points in X giving orbits that are dense on each n-th projection in the form of a countable intersection of open sets. Show each set is dense using the transitivity result in finite dimensions. Use Baire Category Theorem to conclude the intersection is dense, hence nonempty. Then show that if an orbit is dense on the projections, it is dense in $X \times \mathbb{R}^{\omega}$.

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Proposition

Cocycles to $\mathbb{R}^\omega$, or ℓ^p, or the C^k subspaces of those, are never stably transitive.

This is because the projections $\pi_n(f)$ can approximate f arbitrarily well in each of these norms but are never transitive.

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- T. Bousch, La condition de Walters. *Ann. Sci. École Norm. Sup.* **34**(2) (2001), 287-311.
- A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.
- I. Melbourne, V. Nitica, and A. Török, Transitivity of Euclidean-type extensions of hyperbolic systems. *Ergodic Theory and Dynamical Systems* **29** (2009), 1585-1602.
- V. Nitica, A note about topologically transitive cylindrical cascades. *Israel J. Math.* **126** (2001), 141-156.
- V. Nitica and M. Pollicott, Transitivity of Euclidean extensions of Anosov diffeomorphisms. *Ergodic Theory and Dynamical Systems*, **25** (2005), 257-269.